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Oscillating Singularities on Cantor Sets: A Grand-Canonical Multifractal Formalism

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The singular behavior of functions is generally characterized by their Hölder exponent. However, we show that this exponent poorly characterizes oscillating singularities. We thus introduce a second exponent that accounts for the oscillations of a singular behavior and we give a characterization of this exponent using the wavelet transform. We then elaborate on a "grand-canonical" multifractal formalism that describes statistically the fluctuations of both the Hölder and the oscillation exponents. We prove that this formalism allows us to recover the generalized singularity spectrum of a large class of fractal functions involving oscillating singularities.

KEY WORDS: Grand-canonical multifractal formalism; invariant measures; fractal functions; cusp singularities; oscillating singularities; Hölder exponent; oscillation exponent; singularity spectrum; wavelet analysis; wavelet transform; modulus maxima; minimizing sequences.

1. INTRODUCTION

During the past few years, there has been increasing interest in the study of irregular objects.[1-3] In order to characterize locally the irregularity of an object, one generally uses the notion of Hölder exponent.[1] Indeed, this exponent can be seen as a measurement of the strength of the singularity behavior of a given function $f(x)$ around a given point $x = x_*$. It is defined as the greatest exponent h so that f is *Lipschitz* h at x_*. This exponent is

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generally denoted $h(x_*)$. Let us recall that f is said to be Lipschitz h at x_* if and only if there exists a constant C and a polynomial $P(x)$ of order smaller than h so that, for all x in a neighborhood of x_*,

$$|f(x) - P(x - x_*)| \leqslant C\,|x - x_*|^h \tag{1}$$

If f is n times continuously differentiable at the point x_*, then one can use for the polynomial $P(x - x_*)$ the order-n Taylor series of f at x_* and thus prove that $h(x_*) > n$. Thus, the Hölder exponent $h(x_*)$ measures how irregular f is at the point x_*. The higher the exponent $h(x_*)$, the more regular the function f. In that sense, one can expect the Hölder exponent at x_* of the primitive of f to be greater than the one of f. Actually, when the singular behavior corresponds to a *cusp* (see Section 2), e.g., the singular part of f in a neighborhood of x_* is of the order of $|x - x_*|^h$ [consequently, we get $h(x_*) = h$], the primitive of f has exactly the Hölder exponent $h(x_*) + 1$. In that case, the numerical estimation of the Hölder exponent $h(x_*)$ is rather simple. One chooses a function $\psi(x)$ which is well localized around $x = 0$ and which is orthogonal to all the polynomials $P(x)$ up to the order $N > h(x_*)$ so that when one integrates both sides of Eq. (1) against $\psi((x - x_*)/a)$, one gets

$$T_\psi[f](x_*, a) = \frac{1}{a} \int \psi\left(\frac{x - x_*}{a}\right) f(x)\, dx \sim a^{h(x_*)}, \qquad a \to 0^+ \tag{2}$$

The function $T_\psi[f]$, considered as a function of the position x_* and the scale a, is called the *wavelet transform*[4-15] of f. The Hölder exponent $h(x_*)$ can thus be obtained by estimating the power-law behavior of the wavelet transform at the position x_* when varying the scale a.[16-18] However, in the case where f is made up of an accumulation of singular behavior (which is the case if f is a fractal function), the direct estimation of $h(x_*)$ through Eq. (2) is very unstable due to the influence of the singularities in the neighborhood of x_*.[19-23] Thus, in order to estimate the singularity spectrum $D(h)$ of a singular function f [i.e., the Hausdorff dimension of the set of points x corresponding to the same Hölder exponent $h(x) = h$], one cannot just make local measurements of the Hölder exponents h. The multifractal formalism originally introduced in refs. 24–27 and revisited in refs. 28–30 provides a "global" method for estimating this singularity spectrum that is based on the computation of a partition function of the type

$$\mathscr{Z}(q, a) = \sum_i |T_\psi[f](x_i(a), a)|^q \tag{3}$$

There exist different ways of choosing,[23, 28–31] at each scale a, the points $\{x_i(a)\}_i$, leading to different definitions of $\mathscr{Z}(q, a)$. Let us mention one of them,[28–30] which consists in considering all the local maxima $x_i(a)$ of $|T_\psi[f](x, a)|$ considered as a function of x. One can then prove that, for a large class of fractal functions, $\mathscr{Z}(q, a)$ follows a power-law scaling

$$\mathscr{Z}(q, a) \sim a^{\tau(q)}, \qquad a \to 0^+ \tag{4}$$

and that the so-obtained exponents $\tau(q)$ are related to the $D(h)$ singularity spectrum through the Legendre transform[29, 32]:

$$D(h) = \min_q (hq - \tau(q)) \tag{5}$$

Let us note that there exists a deep analogy that links this formalism with statistical thermodynamics.[26, 33–36] The variables q and $\tau(q)$ play the same role as the inverse of temperature and the free energy in thermodynamics, while the Legendre transform (5) indicates that instead of the energy and the entropy, we have h and $D(h)$ as the thermodynamic variables conjugated to q and $\tau(q)$. Since this so-defined formalism is based on the computation of a partition function, it does not involve any local measurement of the Hölder exponents and thus allows us to get very precise estimates of the singularity spectrum. It has been successfully used for characterizing the scaling properties of a very wide range of fractal measures and fractal functions, including the invariant probability distribution on a strange attractor, the distribution of voltage drops across a random resistor network, the dissipation field and the velocity field of fully developed turbulence, the arborescent morphologies of fractal aggregates, the structural complexity of DNA sequences, etc.[2, 3, 14, 15, 23, 35, 36]

However, even though Eq. (2) is not directly used for estimating the Hölder exponents, it is the cornerstone of the multifractal formalism. As this relation holds only for cusplike singularities, this formalism is not valid if the fractal function f involves other types of singularities. Let us consider the chirp function

$$f(x) = |x - x_*|^h \sin\left(\frac{1}{|x - x_*|^\beta}\right), \qquad h > 0, \quad \beta > 0 \tag{6}$$

This function is singular at $x = x_*$ and its Hölder exponent is $h(x_*) = h$. However, a direct estimate of this exponent using Eq. (2) would lead to a wrong result. Indeed, since the function $f(x)$ is infinitely oscillating around x_*, cancellations appear when this function is integrated against a smooth function, leading to a function more regular than expected. Thus, Eq. (2) would lead to an underestimate of the Hölder exponent. Such a singular behavior is referred to as an *oscillating singularity*.[18, 37, 38] Actually, contrary to functions with cusp singularities, the primitive of the oscillating function in Eq. (6) has a Hölder exponent $h+1+\beta \neq h+1$. Let us note that a cusp can be seen as an oscillating singularity with $\beta=0$. Thus, in order to fully characterize a singular behavior (corresponding to a cusp or to an oscillating singularity), one needs two exponents: the Hölder exponent h and the *oscillation exponent* β. The exponent β characterizes the local power-law divergence of the instantaneous frequency. Thus the classical formalism is not adapted for analyzing singular functions involving other types of singularities than cusps in the sense that for singularities other than cusps (i) the Hölder exponents involved in the so-obtained $D(h)$ singularity spectrum are underestimated and (ii) the Hölder exponent alone does not fully characterize the local behavior of the function.

In this paper, we present a generalized multifractal formalism that is adapted to describe the statistics of both the Hölder exponents h and the oscillation exponents β characterizing the singular behavior involved in a given singular function. More specifically, this new formalism allows us to get the singularity spectrum $D(h, \beta)$ which corresponds to the Hausdorff dimension of the set of points x corresponding to the same Hölder and oscillation exponents, i.e., $h(x)=h$ and $\beta(x)=\beta$. Whereas the partition function used in the classical formalism is indexed by a single parameter (conjugated to the Hölder exponent h), this new description is based on a partition function involving two intensive parameters (associated with the exponents h and β). In that sense, it is the analog of a "grand-canonical" formalism, whereas the classical formalism [Eq. (3)] can be identified with a "canonical" description.[36]

The paper is organized as follows. In Section 2 we give a rigorous definition of what cusps and oscillating singularities are. Moreover, we define, for any type of singular behavior, a new exponent β that characterizes the oscillations (if any) of a function around the singularities. In Section 3 we show that self-similar distributions involve only cusp singularities and we illustrate the classical formalism on this class of distributions. In Section 4 we use the wavelet decomposition to define a new class of fractal distributions that involve accumulations of both cusp and oscillating singularities. We introduce a generalized multifractal formalism that provides a natural method to compute their $D(h, \beta)$ singularity spectrum. We conclude in Section 5.

2. WAVELET ANALYSIS OF SINGULAR BEHAVIOR: CUSP AND OSCILLATING SINGULARITIES

2.1. Defining Cusp and Oscillating Singularities from the Wavelet Transform

The wavelet transform of a real-valued function f according to the *analyzing wavelet* ψ is defined as[4, 5]

$$T_\psi[f](b, a) = \frac{1}{a} \int_{-\infty}^{\infty} \psi\left(\frac{x-b}{a}\right) f(x)\, dx \tag{7}$$

where $a \in \mathbb{R}^{+*}$ and $b \in \mathbb{R}$. Generally ψ is chosen to be well localized in both direct and Fourier spaces, so that T_ψ can be seen as an accurate space–frequency analysis (b is the space parameter, whereas $1/a$ is the frequency parameter). As explained above, in order to detect singular behavior, one has to be blind to possible superimposed smooth behavior [the polynomial P in Eq. (1)], thus one has to choose an analyzing wavelet that is orthogonal to polynomials up to a certain order. For our purpose, we will mainly assume that the first N moments of ψ are vanishing,[16–18, 31] i.e.,

$$\int \psi(x)\, x^k\, dx = 0, \qquad 0 \leqslant k < N \tag{8}$$

Such an analyzing wavelet will be referred to as an *order-N wavelet.*

As briefly explained in the Introduction, the wavelet transform allows us to characterize the Hölder exponent of a cusp singularity. Actually, it is a very powerful tool for characterizing any type of singular behavior (not only cusps). Let us give the main theorem that explains how this tool can be used.[16, 17]

Theorem 1. Let ψ be an order-n wavelet and f a function which is uniformly Lipschitz ε for $\varepsilon > 0$ arbitrarily small. Then:

(a) If f is Lipschitz γ at x_* with $\gamma \leqslant n$, then its wavelet transform satisfies

$$|T_\psi[f](x, a)| = O(a^\gamma + |x - x_*|^\gamma) \tag{9}$$

(b) Conversely, if $\gamma \leqslant n$ and if

$$|T_\psi[f](x, a)| = O\left(a^\gamma + \frac{|x-x_*|^\gamma}{|\ln|x-x_*||}\right) \tag{10}$$

then f is Lipschitz γ at x_*.

Thus the singularity strength h of the function f at the point x_* is directly linked to the way the wavelet transform decreases around x_*. Let us note that the necessary condition (9) is not sufficient for f to be Lipschitz γ. Basically, the difference between Eq. (9) and Eq. (10) is the logarithmic term $\ln|x-x_*|$. From a numerical point of view, such a logarithmic correction is negligible. Let us thus introduce some convenient notations which are blind to such corrections and will allow us to derive a necessary and sufficient condition for f to be at x_* of Hölder regularity h.

Notation 1. Let f and g be two positive functions with $g \to 0$ when $x \to x_*$. We introduce three notations $O_{\log}$, $O_{\log}^{-}$, and $O_{\log}^{=}$ that compare the asymptotic behavior of f with that of g when $x \to x_*$:

- $f = O_{\log}(g) \Leftrightarrow \liminf(\log f/\log g) \geqslant 1$
- $f = O_{\log}^{-}(g) \Leftrightarrow \liminf(\log f/\log g) > 1 \quad [\Leftrightarrow \exists \varepsilon > 0, f = O_{\log}(g^{1+\varepsilon})]$
- $f = O_{\log}^{=}(g) \Leftrightarrow \liminf(\log f/\log g) = 1$

where the lim inf's are taken for $x \to x_*$.

Lemma 1. Let f and g be two positive functions with $g \to 0^+$.

(a) $f = O(g) \Rightarrow f = O_{\log}(g)$

(b) $O_{\log}$ is not sensitive to logarithmic corrections, e.g., $f = O_{\log}(g) \Leftrightarrow f = O_{\log}(g\,|\log(g)|)$

(c) $f = O_{\log}(g) \Leftrightarrow \forall \varepsilon > 0, f = O(g^{1-\varepsilon})$

(d) $h = \sup\{\gamma, f = O(g^{\gamma})\} \Leftrightarrow f = O_{\log}^{=}(g^{h})$

(e) If g_1 is another positive function with $g_1 \to 0^+$, then $h = \sup\{\gamma, f = O(g^{\gamma} + g_1^{\gamma})\} \Leftrightarrow f = O_{\log}^{=}(g^{h} + g_1^{h})$

Propositions (a)–(c) are very easy to prove. Proposition (d) is a direct consequence of proposition (c), and (e) is obtained from (d) by considering separately the two domains $g < g_1$ and $g_1 \leqslant g$.

Let us recall that the Hölder exponent $h(x_*)$ of f at the point x_* is defined as the greatest exponent h so that f is Lipschitz h at x_*, i.e.,

$$h(x_*) = \sup\{h, \exists P(x), \exists C, |f(x) - P(x - x_*)| \leqslant C\,|x - x_*|^{h}\} \tag{11}$$

where $P(x)$ is a polynomial.

By using proposition (e) along with Theorem 1, one easily gets a wavelet characterization of the Hölder exponent h:

Theorem 2. Let ψ be an order-n wavelet and f a function which is uniformly Lipschitz ε for $\varepsilon > 0$ arbitrarily small and is singular at $x = x_*$

[i.e., $h(x_*) \neq \infty$]. Then the Hölder exponent of f at x_* is $h < n$ (i.e., h is the greatest exponent γ so that f is Lipschitz γ at x_*) if and only if

$$|T_\psi[f](x, a)| = O_{\log}^{=}(a^h + |x - x_*|^h) \tag{12}$$

Remark. Equation (11) defines the Hölder exponent of any bounded function f. The last theorem gives a characterization of this exponent, using the wavelet transform, in the case f has a minimum regularity. In the case of a measure μ, one generally defines the Hölder exponent of μ at the point x_* by a relation of the type

$$\liminf_{\varepsilon \to 0} \frac{\log \mu(B(x_*, \varepsilon))}{\log \varepsilon} = h(x_*) \tag{13}$$

where $B(x_*, \varepsilon)$ denotes a ball centered at x_* and of size ε. In the case the Hölder exponent of μ satisfies $0 < h(x_*) < 1$, one can easily prove that $h(x_*)$ is also the Hölder exponent of the characteristic function f_μ of μ. Let us note that this is no longer true if $h(x_*) = 1$. Indeed, the Lebesgue measure corresponds to $h(x_*) = 1$, whereas its characteristic function $f_\mu(x) = x$ is not singular and thus corresponds to $h(x_*) = \infty \neq 1$. Actually, the definition (13) does not characterize the *regularity* of the measure μ around x_*. It just characterizes the way the mass scales around x_*. Since in this article we are interested in characterizing the regularity of an object, we will define the Hölder exponent of a measure μ as the Hölder exponent of its characteristic function. Thus, for example, we will say that the Hölder exponent of the Lebesgue measure is $h(x_*) = \infty$ for all x_*. It is easy to prove that if μ has a minimum regularity [i.e., there exist $\varepsilon > 0$ and $C > 0$ so that for all intervals I, $\mu(I) \leqslant C\,|I|^\varepsilon$], then the characterization (12) still holds when replacing f by μ, i.e., the Hölder exponent of μ at x_* is $h < n$ if and only if

$$|T_\psi[\mu](x, a)| = O_{\log}^{=}(a^h + |x - x_*|^h) \tag{14}$$

where the wavelet transform of a measure is defined by

$$T_\psi[\mu](b, a) = \int_{-\infty}^{\infty} \psi\left(\frac{x-b}{a}\right) d\mu \tag{15}$$

Equation (12) in Theorem 2 suggests that we distinguish two types of singular behavior corresponding to the cases where the strongest wavelet coefficients are localized either inside a "cone" $|x - x_*| = O_{\log}(a)$ or outside such a cone. By the term "strongest coefficients" we mean any sequence (x_n, a_n) in the space–scale half-plane that converges toward $(x_*, 0)$ and for which the lim inf corresponding to Eq. (12) is reached. As it corresponds to a sequence that minimizes a certain quantity, we will call it a *minimizing sequence.*

Notation 2. A sequence (x_n, a_n) in the space–scale half-plane that converges toward $(x_*, 0)$ and for which

$$\lim_{n \to \infty} \frac{\log(|T_\psi[f](x_n, a_n)|)}{\log(a_n^{h(x_*)} + |x_n - x_*|^{h(x_*)})} = 1 \tag{16}$$

will be called a *minimizing sequence.*

When the singularity corresponds to a chirp $f(x) = |x - x_*|^h \sin(1/|x - x_*|^\beta)$ ($\beta > 0$), the strongest wavelet coefficients are localized on a "ridge"[39] of the form $a = C\,|x - x_*|^{\beta + 1}$ (where C is a constant) that describes the variation of the instantaneous period. Since the size a_n of an oscillation is much smaller than its distance from x_* $[a_n = C\,|x_n - x_*|^{\beta + 1} = O^-_{\log}(|x_n - x_*|)]$, chirps correspond to the case where the strongest coefficients are outside any cone. In the same way, one easily checks that singularities of the type $f(x) = |x - x_*|^h$ correspond to the case where the strongest coefficients are inside a cone, i.e., $|x_n - x_*| = O_{\log}(a_n)$. Actually, the intuitive remarks above can be used for defining rigorously what cusp or oscillating singularities are.

Definition 1. A function $f(x)$ is said to have a *cusp singularity* at the point x_* if and only if there exists a minimizing series (x_n, a_n) such that

$$|x_n - x_*| = O_{\log}(a_n) \tag{17}$$

Conversely:

Definition 2. A function $f(x)$ is said to have an *oscillating singularity* at the point x_* if and only if it is not a cusp singularity, i.e., for all minimizing series (x_n, a_n) we have

$$a_n = O^-_{\log}(|x_n - x_*|) \tag{18}$$

Remark. Any singularity x_* corresponds either to a cusp singularity or to an oscillating singularity.

2.2. Introducing the Oscillation Exponent β

As explained in the Introduction, oscillating singularities[18, 37, 38] are not fully characterized by their Hölder exponent. Indeed, in the case of a chirp, $f(x) = |x - x_*|^h \sin(1/|x - x_*|^\beta)$, the Hölder exponent h at x_* does not characterize how the instantaneous frequency goes to ∞ when x goes to x_*. Ideally, we would like to have access to both h and β. Actually, as mentioned in the introduction, β plays a very important role in the regularity of f when it is integrated. Indeed, it is very easy to prove that if f is a (h, β) chirp [Eq. (6)], then the singular part of the primitive of f corresponds to an $(h+1+\beta, \beta)$ chirp. Thus, whereas the Hölder exponent increases by 1 for a cusp singularity, it increases by $1+\beta$ for a chirp. More generally, if we fractionally ε-integrate f around x_*,[40] it will increase respectively by ε or by $\varepsilon(1+\beta)$. In that sense, a cusp can be seen as a chirp with $\beta=0$. These remarks can be used for defining, in a general case, an exponent β that will characterize the oscillations of a given singular behavior.

Definition 3. The function f is singular at x_* with the exponents (h, β) if and only if h is the Hölder exponent at x_* and $\beta = (dh_\varepsilon/d\varepsilon)(\varepsilon=0^+)-1$, where h_ε is the Hölder exponent of the ε-primitive f_ε of f at x_*.

Remark. This definition uses the fact that h_ε is right differentiable at $\varepsilon=0$. This will be stated in the next theorem.

Remark. Instead of this definition, one could define the exponent β as the regularity rate that appears when f is integrated a great number of times,[37] i.e., $\beta=\lim_{n\to\infty}(h_n/n)-1$. However, in this case, the value of β becomes very unstable, e.g., the function $f(x)=|x|\sin(1/x)+|x|^\gamma$ $(\gamma \gg 1)$ would correspond to a singular behavior $(x_*=0)$ with exponents $(1, 0)$ (i.e., a cusp of Hölder 1) and not $(1, 1)$ as obtained if the second term is neglected.

The following theorem proved in ref. 40 shows that the definition of the exponent β is consistent and allows us to distinguish between cusps and oscillating singularities. It also gives a characterization of β in terms of minimizing sequences.

Theorem 3. Let f be a function that is singular at x_* [i.e., $h(x_*)\neq\infty$]. Let h_ε be the Hölder exponent of the ε-primitive f_ε of f (with $\varepsilon>0$). The function h_ε is concave and differentiable for all $\varepsilon>0$ and is right differentiable at $\varepsilon=0$. Moreover, the three following assertions hold:

(a) x_* corresponds to a cusp singularity $\Rightarrow \beta=h_0'-1=0$.

(b) x_* corresponds to an oscillating singularity $\Rightarrow \beta=h_0'-1>0$.

(c) In all cases,

$$\beta = h_0' - 1 = \max(0, \liminf \log(a_n)/\log|x_n - x_*| - 1) \tag{19}$$

where the lim inf is taken over all the minimizing sequences (x_n, a_n) when n goes to infinity and where h'_0 denotes the right derivative of h_ε at $\varepsilon = 0$.

The last theorem can be rewritten in a more synthetic form that clearly shows that the exponents (h, β) fully characterize any singularities and that β can be recovered in two different ways (i.e., from the derivation of h_ε or from the minimizing sequences).

Theorem 4. Let $f(x)$ be a function that is singular at $x = x_*$ with the singularity exponents $(h(x_*), \beta(x_*))$ [where $\beta(x_*)$ is defined as in Definition 3]. Then:

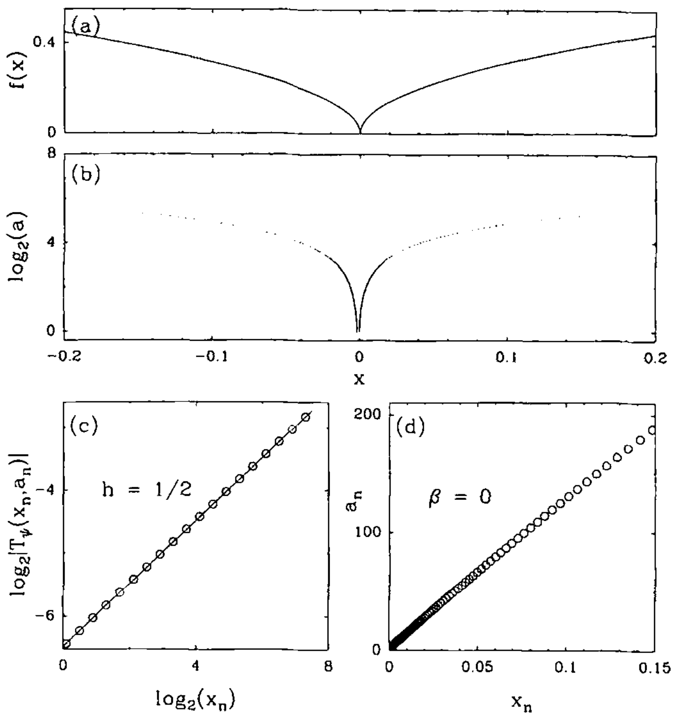

Fig. 1. Detection of the local exponents $h(x_*)$ and $\beta(x_*)$ associated to a cusp singularity. (a) Graph of the function $f(x) = |x|^{1/2}$; the point $x_* = 0$ corresponds to a cusp singularity of f. (b) Wavelet transform skeleton showing the positions of the wavelet transform modulus maxima for the signal in (a). The analyzing wavelet ψ is the first derivative of the Gaussian function. These maxima fall on two maxima lines lying inside a cone $|x - x_*| = O_{\log}(a)$. (c) Plot of $\log_2 |T_\psi[f](x_n, a_n)|$ vs. $\log_2(x_n)$; the slope provides an estimate of $h(x_*) = 1/2$. (d) Plot of a_n vs. x_n; the fact that the points fall on a straight line indicates that $\beta(x_*) = 0$. In (c) and (d) the set of points (x_n, a_n) defining a minimizing sequence corresponds to either one of the two maxima lines illustrated in (b).

(a) x_* corresponds to a cusp singularity $\Leftrightarrow \beta(x_*) = 0$.

(b) x_* corresponds to an oscillating singularity $\Leftrightarrow \beta(x_*) > 0$.

(c) $\beta(x_*) = \max(0, \liminf \log(a_n)/\log |x_n - x_*| - 1)$, where the lim inf is taken over all minimizing sequences [Eq. (16)].

Remark. This last theorem proves that the exponent β that characterizes the variation of the Hölder exponent when f is fractionally integrated can be recovered from minimizing sequences by studying the power-law behavior of the scale a_n versus the distance to the singularity $|x_n - x_*|$. Conversely, one could have defined another exponent β from the Hölder regularity of the ε-derivative of f (if it exists).

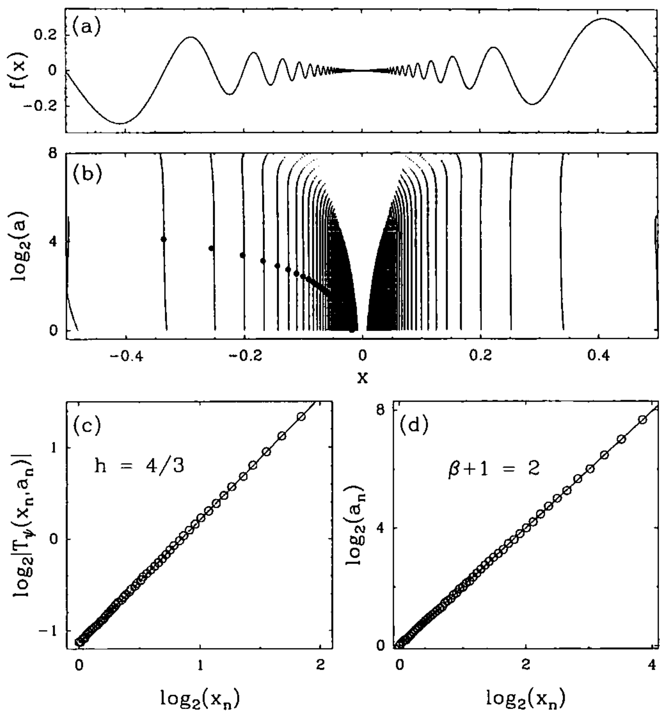

Fig. 2. Detection of the local exponents $h(x_*)$ and $\beta(x_*)$ associated to an oscillating singularity. (a) Graph of the function $f(x) = |x|^{4/3} \sin(2\pi/x)$; the point $x_* = 0$ corresponds to an oscillating singularity of f. (b) Wavelet transform skeleton showing the positions of the wavelet transform modulus maxima for the signal in (a). The analyzing wavelet ψ is the first derivative of the Gaussian function. The maxima lie on maxima lines. Along each line l_n, the dot marks the point (x_n, a_n) where $|T_\psi[f]|$ is the greatest. The set of such points defines a minimizing sequence lying outside any cone. (c) Plot of $\log_2 |T_\psi[f](x_n, a_n)|$ vs. $\log_2(x_n)$; the slopes provides an estimate of $h(x_*) = 4/3$. (d) Plot of $\log_2(a_n)$ vs. $\log_2(x_n)$; the slope gives an estimate of $\beta(x_*) + 1 = 2$, i.e., $\beta(x_*) = 1$.

The "classical" multifractal formalism accounts for the distribution of the Hölder exponents h only,[23, 25, 36] but it leads to wrong results if oscillating singularities ($\beta \neq 0$) are involved in the studied fractal distribution. As we have seen, h gives a poor characterization of a singular behavior; thus the exponent β appears to be essential. Figures 1 and 2 show, respectively, a cusp and an oscillating singularity together with the numerical estimation of the corresponding h and β exponents using the wavelet transform. The goal of this paper is to build a new formalism that accounts for the fluctuations of both h and β. Before presenting this new formalism, let us show that self-similar distributions only involve cusp singularities that can be described by the so-called "canonical" multifractal formalism.[23–36]

3. MULTIFRACTAL FORMALISM FOR SELF-SIMILAR DISTRIBUTIONS

A distribution is self-similar if it is invariant under specific transformations involving mainly dilations and translations. In this section, we study the local and statistical regularity properties of self-similar distributions using their wavelet transforms. For our purpose, we will restrict ourselves to the class of (Bernoulli) measures invariant under piecewise-linear dynamical systems. However, all the results reported below remain valid for more general self-similar distributions associated to hyperbolic mappings.[29, 32]

3.1. The Dynamical System in the Wavelet Transform Half-Plane

Let us consider the expanding piecewise-linear map T on $A = [0, 1]$ for which $T^{-1}(A)$ is a finite union of disjoint intervals

$$T^{-1}(A) = \bigcup_{i=1}^{s} A_i \tag{20}$$

We suppose that the smallest *gap* between two consecutive intervals is strictly positive, i.e., $\min_i\{\mathrm{dist}(A_i, A_{i+1})\} > 0$. We then define

$$\begin{aligned} T_i^{-1}\colon\ & A \to A_i \\ & x \to T_i^{-1}(x) = T^{-1}(x) = v_i x + x_i \end{aligned} \tag{21}$$

where T is assumed to be hyperbolic, i.e., $0 < v_i < 1$. Let us define the n-cylinders $A_{k_1 \cdots k_n}$:

$$A_{k_1 \cdots k_n} = A \cap T_{k_1}^{-1}(A) \cap (T_{k_1} \circ T_{k_2})^{-1}(A) \cdots \cap (T_{k_1} \circ T_{k_2} \circ \cdots \circ T_{k_n})^{-1}(A) \tag{22}$$

Then if J denotes the invariant set under the mapping T, J is the limit of the set (when $n \to +\infty$)

$$\mathscr{A}^n = A \cap T^{-1}(A) \cdots \cap T^{-n}(A) = \bigcup_{\substack{k_i = 1 \cdots s \\ i = 1 \cdots n}} A_{k_1 \cdots k_n} \tag{23}$$

i.e., J can be written as

$$J = \bigcap_{n}^{\infty} \mathscr{A}^{(n)} \tag{24}$$

Thus any point x_* in J can be adressed in a unique way through a "kneading sequence" $k_1 k_2 \cdots k_n \cdots$ in the sense that $\lim_{n \to \infty} A_{k_1 k_2 \cdots k_n} = x_*$.

The mapping T is a linear version of more general one-dimensional mappings usually referred to as "cookie cutters"[27] or expanding Markov maps.[26] One can associate to this mapping a family of invariant measures (called the Bernoulli measures) for which T is ergodic. A Bernoulli measure is a measure μ which is supported by the set J and which satisfies $\exists (\mu_1, \ldots, \mu_s) \in]0, 1[^s$, $\sum_i \mu_i = 1$, so that

$$\forall (k_1 \cdots k_n) \in \{1, \ldots, s\}^n, \quad \mu(A_{k_1 \cdots k_n}) = \mu_{k_1} \cdots \mu_{k_n} \tag{25}$$

These measures are self-similar in the sense that

$$\mu = \frac{1}{\mu_k} \mu \circ T_k^{-1} \tag{26}$$

If we choose ψ with a compact support, it follows that the wavelet transform of a Bernoulli measure μ satisfies[29, 41, 42]

$$T_\psi[\mu](b, a) = \frac{1}{\mu_k} T_\psi[\mu](T_k^{-1}(b), v_k a), \qquad \forall b \in A \tag{27}$$

for a small enough. This last relation means that the wavelet transform of μ is invariant under the mappings $\tilde{T}_k$,

$$T_\psi[\mu] = \frac{1}{\mu_k} T_\psi[\mu] \circ \tilde{T}_k^{-1} \tag{28}$$

where

$$\tilde{T}_k(b, a) = (T_k(b), a/v_k) \tag{29}$$

Let us consider the point $b = b_0$, where $|T_\psi[\mu](b, a_0)|$ is maximum (where a_0 is a small enough fixed scale). This value corresponds to a point (b_0, a_0) for which $|T_\psi[\mu](b_0, a_0)| > 0$. For the sake of simplicity, in the following, we will consider that $|T_\psi[\mu](b_0, a_0)| = 1$ and $a_0 = 1$. Since the analyzing wavelet ψ is localized around $x = 0$, we can also suppose that $b_0 \in A$. Then, using Eqs. (28) and (29), one can associate to any n-cylinder $A_{k_1 \cdots k_n}$ a point $(b_{k_1 \dots k_n}, a_{k_1 \dots k_n})$ which corresponds to the maximum of the wavelet transform $|T_\psi[\mu](b, a)|$ when $a = a_{k_1 \dots k_n}$ and $b \in A_{k_1 \cdots k_n}$. From Eq. (28), we deduce

$$(b_{k_1 \cdots k_n}, a_{k_1 \cdots k_n}) = \tilde{T}_{k_n}^{-1}(b_{k_1 \cdots k_{n-1}}, a_{k_1 \cdots k_{n-1}}) \tag{30}$$

and

$$T_\psi[\mu](b_{k_1 \dots k_{n-1}}, a_{k_1 \cdots k_{n-1}}) = \frac{1}{\mu_{k_n}} T_\psi[\mu](b_{k_1 \dots k_n}, a_{k_1 \cdots k_n}) \tag{31}$$

Recursively, from the last two equations, we get

$$a_{k_1 k_2 \cdots k_n} = v_{k_1} v_{k_2} \cdots v_{k_n} \tag{32}$$

and

$$T_\psi[\mu](b_{k_1 \cdots k_n}, a_{k_1 \cdots k_n}) = \mu_{k_1} \mu_{k_2} \cdots \mu_{k_n} \tag{33}$$

Let $x_* \in J$ be the point corresponding to the kneading sequence $k_1 \cdots k_n \cdots$. Since the gap between two intervals A_k is strictly positive, it is easy to prove that the sequence $(b_{k_1 \cdots k_n}, a_{k_1 \cdots k_n})_{n \in N}$ contains some minimizing sequences and that the exponent $\beta(x_*)$ is reached for some such sequences. Thus, since $x_* \in A_{k_1 \cdots k_n}$ and $b_{k_1 \cdots k_n} \in A_{k_1 \cdots k_n}$, then $|b_{k_1 \cdots k_n} - x_*| \leqslant |A_{k_1 \cdots k_n}|$, where $|A_{k_1 \cdots k_n}|$ stands for the size of the interval $A_{k_1 \cdots k_n}$. Then, from

$$|A_{k_1 \cdots k_n}| = v_{k_1} \cdots v_{k_n} = a_{k_1 \cdots k_n} \tag{34}$$

one finally gets

$$\beta(x_*) = 0 \tag{35}$$

From Theorem 4, one easily deduces the following proposition.[29, 38]

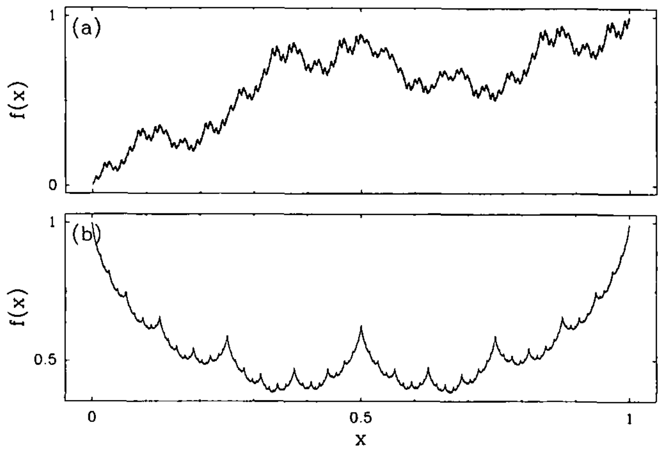

Fig. 3. Fractal distributions belonging to the class $\mathcal{M}$ of singular distributions that involve cusp singularities only. (a) Characteristic function of a signed Bernoulli measure with parameters $\nu_1 = \nu_2 = \nu_3 = \nu_4 = 1/4$ and $\mu_1 = 0.4$, $\mu_2 = 0.5$, $\mu_3 = -0.4$, and $\mu_4 = 0.5$. (b) Homogeneous fractal function constructed iteratively with the weights $\mu_1 = \mu_2 = 0.3$ on a dyadic ($\nu_1 = \nu_2 = 1/2$) order-1 spline wavelet basis (using normalization a^{-1} instead of $a^{-1/2}$).

Proposition 1. The Bernoulli measures are singular measures that involve cusp singularities only.

In the following, we will call $\mathcal{M}$ the class of singular distributions whose wavelet transform maxima (for a particular analyzing wavelet ψ) satisfies the self-similarity relations (30) and (33). Two examples of such distributions are illustrated in Fig. 3. This set is actually much larger than the set of the Bernoulli measures μ defined above.[29, 32]

Remark. We claim in Proposition 1 that singular distributions belonging to $\mathcal{M}$ do not contain oscillating singularities. This result can be easily extended to self-similar distributions that are invariant under hyperbolic dynamical systems. However, we have shown in ref. 38 that the lack of hyperbolicity of the dynamical system implies the occurrence of an infinite number of oscillating singularities. For example, the chirp function $f(x) = \sin(2\pi/x)$ is invariant under the mapping $T(x) = x/(1-x)$, which is nonhyperbolic (marginally dilating) at the origin $x_* = 0$.

3.2. Extracting the $D(h)$ Singularity Spectrum Using the Multifractal Formalism

In this section, we consider a distribution $\mu \in \mathcal{M}$. This means that we suppose that there exists an analyzing wavelet ψ and some weights

$(\mu_k)_{k\in\{1,\ldots,s\}}$ $(0<\mu_k<1,\ \forall k)$ such that the wavelet transform maxima of μ satisfy

$$(b_{k_1\cdots k_n}, a_{k_1\cdots k_n}) = \tilde{T}_{k_n}^{-1}(b_{k_1\cdots k_{n-1}}, a_{k_1\cdots k_{n-1}}) \tag{36}$$

and

$$T_\psi[\mu](b_{k_1\cdots k_n}, a_{k_1\cdots k_n}) = \mu_{k_1}\cdots\mu_{k_n} \tag{37}$$

The goal of the multifractal formalism is to give a method for computing the $D(h)$ singularity spectrum of μ. Let us recall that the $D(h)$ singularity spectrum of a singular distribution μ is defined as the Hausdorff dimension of the set of all the points x corresponding to the same Hölder exponent h, i.e.,

$$D(h) = \mathrm{Dim}_H\{x, h(x) = h\} \tag{38}$$

At step n, we cover the support of the singularity of μ with the covering $\mathscr{A}^{(n)}$, i.e., with all the n-cylinders $(A_{k_1\cdots k_n})$, and we define the following partition function:

$$\mathscr{Z}_n(q,\tau) = \sum_{A_{k_1\cdots k_n}} \Big(\sup_{b\in A_{k_1\cdots k_n}} |T_\psi[\mu](b, a_{k_1\cdots k_n})| \Big)^q a_{k_1\cdots k_n}^{-\tau} \tag{39}$$

where q and τ are real numbers. From the definition of the maxima $(b_{k_1\cdots k_n}, a_{k_1\cdots k_n})$, this partition function can be rewritten as

$$\mathscr{Z}_n(q,\tau) = \sum_{(k_1\cdots k_n)} |T_\psi[\mu](b_{k_1\cdots k_n}, a_{k_1\cdots k_n})|^q a_{k_1\cdots k_n}^{-\tau} \tag{40}$$

Let us define the exponent $\alpha(x_*)$ (where $x_*\in J$ corresponds to the kneading sequence $k_1\cdots k_n\cdots$) as follows:

$$\alpha(x_*) = \liminf_{n\to\infty} \frac{\log |T_\psi[\mu](b_{k_1\cdots k_n}, a_{k_1\cdots k_n})|}{\log a_{k_1\cdots k_n}} \tag{41}$$

Let us also define the spectrum $F(\alpha)$:

$$F(\alpha) = \mathrm{Dim}_H\{x, \alpha(x) = \alpha\} \tag{42}$$

Using the thermodynamic analogy, the energy corresponds to α, which is conjugate to the inverse of the temperature q. The main statement of the multifractal formalism is that the entropy basically corresponds to the singularity spectrum associated to the exponent α, i.e., $F(\alpha) = \mathrm{Dim}_H\{x, \alpha(x) = \alpha\}$. Since all the singularities involved in μ are cusps, the exponent $\alpha(x_*)$ and the Hölder exponent $h(x_*)$ are equal and consequently $F(\alpha)=D(h=\alpha)$. The entropy thus corresponds to the $D(h)$ singularity spectrum. Moreover, $\alpha(x_*)$ can be equally defined as the lim inf of $\log |T_\psi[\mu](b_{k_1\cdots k_n}, a_{k_1\cdots k_n})|/\log a_{k_1\cdots k_n}$ [as in Eq. (41)] or the lim sup of the same quantity or any other value between the lim inf or the lim sup; this does not change the function $F(\alpha)$. The following theorem gives a rigorous version of this statement.

Theorem 5. Let $\mu\in\mathscr{M}$. Let $\mathscr{Z}_n(q,\tau)$ be its corresponding partition function defined in Eq. (40). Let $\alpha(x)$ be a function on J that satisfies

$$\liminf_{n\to\infty}\frac{\log |T_\psi[\mu](b_{k_1\cdots k_n}, a_{k_1\cdots k_n})|}{\log a_{k_1\cdots k_n}} \leqslant \alpha(x_*)\leqslant \limsup_{n\to\infty}\frac{\log |T_\psi[\mu](b_{k_1\cdots k_n}, a_{k_1\cdots k_n})|}{\log a_{k_1\cdots k_n}} \tag{43}$$

Then, $\forall q\in\mathbb{R}$, there exists a transition exponent $\tau(q)$ such that

$$\tau<\tau(q)\Rightarrow \lim_{n\to\infty}\mathscr{Z}_n(q,\tau)=0$$

$$\tau>\tau(q)\Rightarrow \lim_{n\to\infty}\mathscr{Z}_n(q,\tau)=+\infty$$

The exponent $\tau(q)$ is characterized by the relation

$$Z_1(q,\tau(q))=\sum_{k=1}^{k=s}\mu_k^q v_k^{-\tau(q)}=1 \tag{44}$$

Moreover, the spectrum $F(\alpha)$ of the exponent α [defined by Eq. (42)] does not depend on the choice of the function $\alpha(x)$ satisfying (43). $F(\alpha)$ [and consequently $D(h)$] is obtained by Legendre transforming $\tau(q)$:

$$F(\alpha)=D(h=\alpha)=\min_q(\alpha q-\tau(q)) \tag{45}$$

Proof. For the proof of this theorem we refer the reader to refs. 26, 27, 29, and 32. ∎

As an illustration, the $\tau(q)$ and $D(h)$ spectra of a multifractal function that belongs to the class $\mathscr{M}$ are shown in Fig. 4.

Remark. Let us note that this theorem is based on a partition function $\mathscr{Z}_n(q,\tau)$ defined at each step n of the construction process. Thus, this formulation cannot be used numerically if the construction process is not

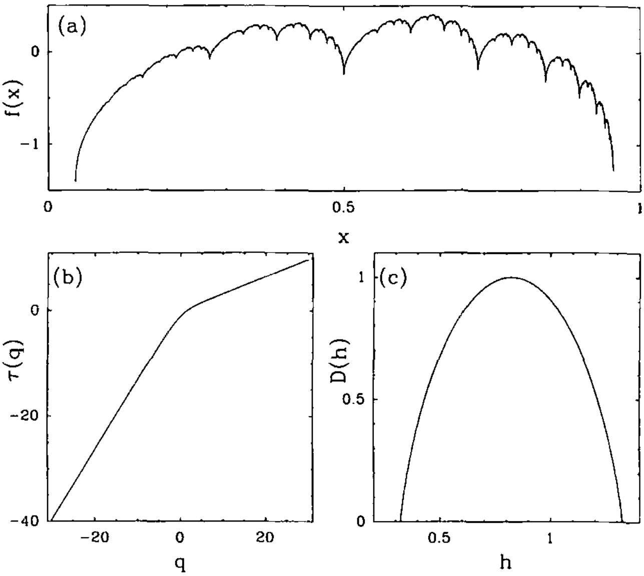

Fig. 4. Multifractal spectra of a nonhomogeneous distribution that belongs to the class $\mathcal{M}$. (a) The graph of the function $-f(x)$, where f was constructed iteratively with the weights $\mu_1 = 0.2$, $\mu_2 = 0.4$ on a dyadic ($\nu_1 = \nu_2 = 1/2$) order-1 spline wavelet basis (using normalization a^{-1} instead of $a^{-1/2}$). (b) $\tau(q)$ vs. q. (c) $D(h)$ vs. h.

known *a priori*. Actually, there exists a version of this theorem that relies on a scale-based partition function that is defined at each scale a from the wavelet coefficients [such as in Eq. (3)]. The wavelet transform modulus maxima (WTMM) method introduced in refs. 28–30 is an implementation of this version which provides a very efficient way of computing the singularity spectrum of a given singular object. We refer the reader to refs. 14, 21, 23, 36, 43, and 44 for more details and specific applications to experimental situations, e.g., fully developed turbulence data, DNA walks, etc.

4. THE GENERALIZED MULTIFRACTAL FORMALISM FOR DISTRIBUTIONS INVOLVING BOTH CUSP AND OSCILLATING SINGULARITIES

4.1. The Dynamical System in the Wavelet Transform Half-Plane

As we have shown in the previous section, the singular distributions in $\mathcal{M}$ involve cusp singularities only. This comes from the fact that the same ratio ν_k is used in the $\tilde{T}_k$'s for the space parameter b and the scale parameter a [Eq. (34)]. A simple way of building a large class of distributions that involve not only cusps but also oscillating singularities consists in using two different ratios in the $\tilde{T}_k$'s. Let us call $\mathcal{N}$ the set of all the fractal distributions f whose wavelet transform maxima satisfy

$$(b_{k_1 \cdots k_n}, a_{k_1 \cdots k_n}) = \hat{T}_{k_n}^{-1}(b_{k_1 \cdots k_{n-1}}, a_{k_1 \cdots k_{n-1}}) \tag{46}$$

with

$$\hat{T}_k(b, a) = (T_k(b), a/\lambda_k) \tag{47}$$

where $0 < \lambda_k \leqslant \nu_k$. Moreover, let us impose that

$$T_\psi[f](b_{k_1 \cdots k_n}, a_{k_1 \cdots k_n}) = \mu_{k_1} \cdots \mu_{k_n} \tag{48}$$

and that $b_0 \notin J$. Let us recall that b_0 is the position of the maximum of the wavelet transform at scale $a_0 = 1$. Along with Eq. (46), it defines the position of all the maxima points at scales $a_{k_1 \cdots k_n}$.

Let $x_* \in J$ be the point corresponding to the kneading sequence $k_1 \cdots k_n \cdots$. As we have already pointed out for the distributions in $\mathcal{M}$, the sequence $(b_{k_1 \cdots k_n}, a_{k_1 \cdots k_n})_{n \in N}$ once again contains all the minimizing sequences. This result is easily deduced from the following lemma.

Lemma 2. Let $x_* \in J$ be the point corresponding to the kneading sequence $k_1 \cdots k_n \cdots$. Then the distance from the maxima point $b_{k_1 \cdots k_n}$ to x_* behaves like

$$|b_{k_1 \cdots k_n} - x_*| \sim |A_{k_1 \cdots k_n}| \tag{49}$$

where $|A_{k_1 \cdots k_n}|$ stands for the size $\nu_{k_1} \cdots \nu_{k_n}$ of the n-cylinder $A_{k_1 \cdots k_n}$. More generally, the distance between x_* and any other maxima point of the form $b_{k_1 \cdots k_n k'_{n+1} \cdots k'_m}$ with $k'_{n+1} \neq k_{n+1}$ behaves in the same way,

$$|b_{k_1 \cdots k_n k'_{n+1} \cdots k'_m} - x_*| \sim |b_{k_1 \cdots k_n} - x_*| \sim |A_{k_1 \cdots k_n}| \tag{50}$$

This lemma follows from the self-similarity properties of the maxima points and the fact that $b_0 \notin J$.[40]

From this lemma and the fact that $\lambda_k \leqslant \nu_k$ ($\forall k$), one deduces that there exists a constant C such that

$$|b_{k'_1 \cdots k'_m} - x_*| \geqslant C a_{k'_1 \cdots k'_m}$$

for any sequence $k'_1 \cdots k'_m$. Thus, using Eq. (16), one gets that a minimizing sequence (b_n, a_n) is a sequence that minimizes the quantity $\log(T_\psi[f](b_n, a_n))/\log(|b_n - x_*|)$ when $n \to \infty$. Since

$$T_\psi[f](b_{k_1 \cdots k_n k'_{n+1} \cdots k'_m}, a_{k_1 \cdots k_n k'_{n+1} \cdots k'_m}) \leqslant T_\psi[f](b_{k_1 \cdots k_n}, a_{k_1 \cdots k_n})$$

we get, using Eq. (50) $(k'_{n+1} \neq k_{n+1})$,

$$\frac{\log(T_\psi[f](b_{k_1 \cdots k_n k'_{n+1} \cdots k'_m}, a_{k_1 \cdots k_n k'_{n+1} \cdots k'_m}))}{\log(|b_{k_1 \cdots k_n k'_{n+1} \cdots k'_m} - x_*|)} \geqslant \frac{\log(T_\psi[f](b_{k_1 \cdots k_n}, a_{k_1 \cdots k_n}))}{\log(|b_{k_1 \cdots k_n} - x_*|)}$$

Thus, all the minimizing sequences are subsequences of the sequence $(b_{k_1 \cdots k_n}, a_{k_1 \cdots k_n})_{n \in N}$. On the other hand,

$$|b_{k_1 \cdots k_n} - x_*| \sim |A_{k_1 \cdots k_n}| = \nu_{k_1} \cdots \nu_{k_n} \geqslant \lambda_{k_1} \cdots \lambda_{k_n} = a_{k_1 \cdots k_n} \tag{51}$$

Thus, if there exists one k such that $\lambda_k \neq \nu_k$, then there exist points x_* for which $a_{k_1 \cdots k_n} = O^-_{\log}(|A_{k_1 \cdots k_n}|)$, i.e., points x_* that correspond to oscillating singularities, and consequently the distribution f belongs to $\mathcal{N}$, but not to $\mathcal{M}$. Moreover, if f involves some cusps (e.g., $f \in \mathcal{M}$), they necessarily correspond the case

$$a_{k_1 \cdots k_n} = O^=_{\log}(|b_{k_1 \cdots k_n} - x_*|) \tag{52}$$

We can thus state the following proposition:

Proposition 2. Let $f \in \mathcal{N}$ and $x_* \in J$ be the point that corresponds to the kneading sequence $k_1 \cdots k_n \cdots$. All the minimizing sequences of f associated to x_* are subsequences of the sequence $(b_{k_1 \cdots k_n}, a_{k_1 \cdots k_n})_{n \in N}$. Moreover, if $f \notin \mathcal{M}$, it involves some oscillating singularities.

Remark. The simplest way of building a distribution f which belongs to $\mathcal{N}$ but not to $\mathcal{M}$ is to write f as a sum of wavelets corresponding to the position $b_{k_1 \cdots k_n}$, the size $a_{k_1 \cdots k_n}$, and the amplitude $\mu_{k_1 \cdots k_n}$ of the wavelet transform maxima:

$$f(x) = \sum_{n>0} \sum_{k_1 \cdots k_n} \mu_{k_1} \cdots \mu_{k_n} \psi\left(\frac{x - b_{k_1 \cdots k_n}}{a_{k_1 \cdots k_n}}\right) \tag{53}$$

Actually, if one does so, then one can prove that Eqs. (46) and (48) hold only if one assumes that ψ is an orthonormal wavelet and ν_k and λ_k are some integer powers of 1/2. In this case, the points $(b_{k_1 \cdots k_n}, a_{k_1 \cdots k_n})$ no longer exactly correspond to the maxima of the continuous wavelet transform, but they correspond to the only orthonormal wavelet coefficients that are different from 0. Nevertheless, the necessary and sufficient condition for a distribution to be Hölder h [Eq. (12)] is the same whether one uses the continuous or the orthonormal wavelet transform, provided one replaces the maxima by the orthogonal coefficients. In the case that ν_k and λ_k are not integer powers of 1/2, one thus needs to adjust the value of these parameters in order to match the dyadic grid of the orthogonal wavelet transform. For the sake of simplicity, we will suppose that these parameters are integer powers of 1/2 (the case where they are not can be treated in the same way) and that Eqs. (46) and (48) hold. The rigorous construction of distributions in $\mathcal{N}$ using Eq. (53) with an orthonormal wavelet basis is fully described in ref. 40. Two examples of functions belonging to $\mathcal{N}$ and not to $\mathcal{M}$ are illustrated in Fig. 5.

Let us note that the functions in $\mathcal{N}$ do not correspond to what we referred to a self-similar functions in Section 3. Indeed, the self-similarity properties of such functions cannot be expressed by a simple relation such as Eq. (26). When zooming in such a function f around a given point x and rescaling the values of f, one no longer obtains the "same" function f. The structure looks the same in the sense that one can recognize all the details, but the sizes of these details have to be rescaled in the right way in order to recover f. This type of property can be easily expressed using the wavelet transform, since it allows us to address separately the size of the details (with the scale parameter a) and the position of these details (with the space parameter b).

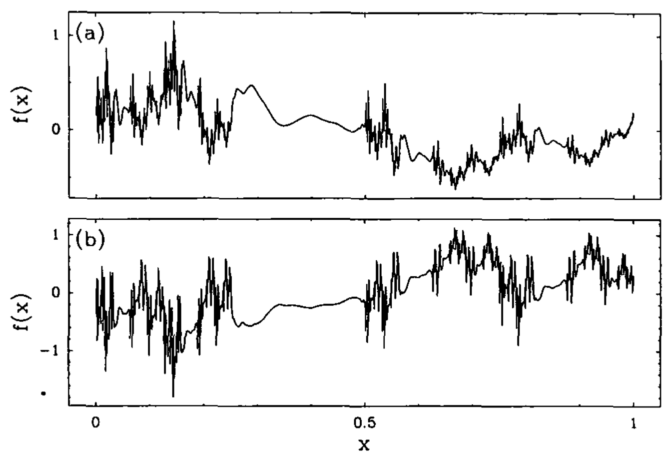

Fig. 5. Fractal distributions belonging to the class $\mathcal{N}$ of singular distributions that involve oscillating singularities. These functions were iteratively constructed on the Daubechies nine-wavelet basis. (a) $\nu_1 = \nu_2 = \nu_3 = \nu_4 = 1/4$; $\lambda_1 = 1/8$, $\lambda_2 = \lambda_3 = \lambda_4 = 1/4$; $\mu_1 = 0.5$, $\mu_2 = 0$, $\mu_3 = 0.45$, $\mu_4 = 0.3$. (b) Same parameters as in (a), but $\mu_i = (\nu_i)^{1/2} = 1/2$ for $i = 1$–4.

4.2. Computing the $D(h, \beta)$ Singularity Spectrum Using a Grand-Canonical Multifractal Formalism

Let f be a distribution in $\mathcal{N}$. In this section, we build a new multifractal formalism that allows us to compute the $D(h, \beta)$ singularity spectrum:

$$D(h, \beta) = \mathrm{Dim}_H \mathcal{D}_{h,\beta} \tag{54}$$

where $\mathcal{D}_{h,\beta}$ is the set defined by

$$\mathcal{D}_{h,\beta} = \{x, h(x) = h \text{ and } \beta(x) = \beta\} \tag{55}$$

For the sake of simplicity, let us define the exponent $\gamma(x) = \beta(x) + 1$ and the set $\mathcal{F}_{h,\gamma} = \mathcal{D}_{h,\gamma-1}$. Our goal is to compute the Hausdorff dimension $F(h, \gamma) = D(h, \beta = \gamma - 1)$.

As we have seen before for distributions in $\mathcal{M}$, the scaling behavior of the partition function defined in Eq. (39) can be used to estimate the spectrum $D(h)$ defined in Eq. (38). In the case of distributions in $\mathcal{N}$, since both cusp and oscillating singularities are involved, we have to introduce in the partition function another quantity that will be able to account for the fluctuations of the oscillation exponent β. We thus define the new partition function in the following way:

$$\mathcal{Z}_n(q, p, \tau) = \sum_{A_{k_1 \cdots k_n}} \Big(\sup_{b \in A_{k_1 \cdots k_n}} |T_\psi[f](b, a_{k_1 \cdots k_n})| \Big)^q |A_{k_1 \cdots k_n}|^{-\tau} a^p_{k_1 \cdots k_n} \tag{56}$$

where p, q, and τ are three real numbers and $|A_{k_1 \cdots k_n}|$ stands for the size of the n-cylinder $A_{k_1 \cdots k_n}$. Let us note that $|A_{k_1 \cdots k_n}| = \nu_{k_1} \cdots \nu_{k_n}$. As discussed in Section 4.1, $\mathcal{Z}_n(q, p, \tau)$ can be rewritten in the following way:

$$\mathcal{Z}_n(q, p, \tau) = \sum_{(k_1 \cdots k_n)} |T_\psi[f](b_{k_1 \cdots k_n}, a_{k_1 \cdots k_n})|^q |A_{k_1 \cdots k_n}|^{-\tau} a^p_{k_1 \cdots k_n} \tag{57}$$

From Eqs. (46)–(48), we get $|T_\psi[f](b_{k_1 \cdots k_n}, a_{k_1 \cdots k_n})| = \mu_{k_1} \cdots \mu_{k_n}$ and $a_{k_1 \cdots k_n} = \lambda_{k_1} \cdots \lambda_{k_n}$ and thus

$$\mathcal{Z}_n(q, p, \tau) = \sum_{(k_1 \cdots k_n)} |\mu_{k_1} \cdots \mu_{k_n}|^q (\nu_{k_1} \cdots \nu_{k_n})^{-\tau} |\lambda_{k_1} \cdots \lambda_{k_n}|^p$$

It can be factorized in the following way:

$$\mathcal{Z}_n(q, p, \tau) = \left(\sum_{k=1}^{s} \mu_k^q \nu_k^{-\tau} \lambda_k^p \right)^n$$

from which it is easy to prove that the so-defined function $P(q, p, \tau)$,

$$P(q, p, \tau) = \lim_{n \to +\infty} n^{-1} \log \mathscr{Z}_n(q, p, \tau) = \log \left(\sum_{k=1}^{s} \mu_k^q \lambda_k^p v_k^{-\tau} \right) \tag{58}$$

is real analytic and convex in each of its argument and that there exists a real, concave analytic function $\tau(q, p)$ defined by

$$P(q, p, \tau(q, p)) = 0 \tag{59}$$

i.e.,

$$\mathscr{Z}_n(q, p, \tau(q, p)) = 1 \tag{60}$$

Let us prove the following main theorem, which allows us to compute the spectrum $D(h, \beta)$ from the function $\tau(q, p)$:

Theorem 6. Let $f \in \mathcal{N}$ and $\mathscr{Z}_n(q, p, \tau)$ be its corresponding partition function defined in Eq. (57) and $\tau(q, p)$ the transition exponent defined in Eq. (59). Then the singularity spectrum $D(h, \beta)$ of f is the *Legendre transform* of the function

$$D(h, \beta) = \min_{q, p} (qh + p(\beta + 1) - \tau(q, p)) \tag{61}$$

Proof. (a) Let us first get the upper bound in Eq. (61), i.e.,

$$D(h, \beta) = F(h, \gamma) \leqslant \min_{q, p} (qh + p\gamma - \tau(q, p)) \tag{62}$$

In the following, $\forall x \in J$, let $A_n(x)$ be the n-cylinder containing x and let $\mu_n(x)$, $v_n(x)$, and $\lambda_n(x)$ be its measure, size, and scale, respectively.

Let $q = q'l$ and $p = p'l$ with $q' + p' = 1$. We fix q' and p' and we consider only l varying. Let

$$\mu'_i = \mu_i^{q'} \lambda_i^{p'}$$

Then Eq. (60) can be rewritten as

$$\mathscr{Z}_n(q, p, \tau(q, p)) = \sum_{(k_1 \cdots k_n)} |\mu'_{k_1} \cdots \mu'_{k_n}|^l (v_{k_1} \cdots v_{k_n})^{-\tau(q'l, p'l)} = 1$$

This last quantity can be seen as a partition function of the type we introduced in the previous section in Eq. (40). The exponent l plays the role of q, the μ'_k the role of the μ_k. Since for $\tau = \tau(q'l, p'l)$, $\mathscr{Z}_n$ is equal to 1,

$\tau(q'l, p'l)$ must correspond to the critical exponent $\tau(q)$ of Theorem 5. The exponent conjugate to l [$\alpha(x)$ in Theorem 5] can be chosen to be $\delta(x) = q'h(x) + p'\gamma(x)$. Actually, from Lemma 2 in Section 4.1, $\forall (q', p') \in \mathbb{R}^2$ and $x = \lim_{n \to \infty} A_n(x) \in J$, we know that there exists a subsequence $\{n_i\}$ such that

$$\delta(x) = \lim_{i \to \infty} \log(\mu_{n_i}^{q'}(x)\, \lambda_{n_i}^{p'}(x))/\log \nu_{n_i}(x)$$

Thus $\delta(x)$ is a "possible" function introduced in Eq. (43) of Theorem 5. One can then apply this theorem to compute the singularity spectrum associated to the exponent δ:

$$D_{q', p'}(\delta) = \mathrm{Dim}_H(\{x, h(x)\, q' + \gamma(x)\, p' = \delta\}) = \min_l\, (l\delta - \tau(q'l, p'l))$$

Since $\forall (h, \gamma)$,

$$\mathscr{D}_{h, \gamma} = \bigcap_{q'h + p'\gamma = \delta} \{x, h(x)\, q' + \gamma(x)\, p' = \delta\}$$

one then deduces the following inequality:

$$F(h, \gamma) \leqslant \min_{q', p'} D_{q', p'}(\delta) = \min_{q, p}\, (qh + p\gamma - \tau(q, p))$$

i.e.,

$$D(h, \beta) \leqslant \min_{q, p}\, (qh + p(\beta + 1) - \tau(q, p))$$

(b) Let us now prove the reverse inequality,

$$D(h, \beta) = F(h, \gamma) \geqslant \min_{q, p}\, (qh + p\gamma - \tau(q, p)) \tag{63}$$

For that purpose we follow the same line as in ref. 27, where similar results were proven for Gibbs states associated to "cookie-cutters." Let us remark that the partition function $\mathscr{Z}_n(q, p, \tau)$ defined in Eq. (57) can be considered as the partition function associated to the measure

$$\mu_{q, p, \tau}(A_{k_1 \cdots k_n}) = e^{-nP(q, p, \tau)} \mu_{k_1}^q \cdots \mu_{k_n}^q \lambda_{k_1}^p \cdots \lambda_{k_n}^p \nu_{k_1}^{-\tau} \cdots \nu_{k_n}^{-\tau}$$

which is the Gibbs state [associated to the linear dynamical system $T(x)$] of the function

$$\varphi_{q, p, \tau}(x) = q\varphi_\mu(x) + p\varphi_\lambda(x) - \tau\varphi_\nu(x) \tag{64}$$

where $\varphi_\nu(x)$ (resp. φ_λ and φ_μ) is a continuous real function equal to $-\log |dT_i/dx| = \log \nu_i$ (resp. $\log \lambda_i$ and $\log \mu_i$) for $x \in A_i$ ($i \in \{1 \cdots s\}$). The function $P(q, p, \tau)$ defined in Eq. (58) is the *pressure* of this Gibbs state.[27] Let

$$s(\mu_{q,p,\tau}) = \lim_{n \to \infty} -n^{-1} \sum_{k_1 \cdots k_n} \mu_{q,p,\tau}(A_{k_1 \cdots k_n}) \log(\mu_{q,p,\tau}(A_{k_1 \cdots k_n}))$$

be the metric entropy associated to the measure $\mu_{q,p,\tau}$. It is straightforward to recover the well-known fact that the Gibbs state $\mu_{q,p,\tau}$ saturates the *variational principle* inequality[35]:

$$P(q, p, \tau) = s(\mu_{q,p,\tau}) + \int \varphi_{q,p,\tau}(x)\, d\mu_{q,p,\tau} \tag{65}$$

Let us call $\mathscr{S}$ the set of points in the (h, γ) plane defined by $(h, \gamma) \in \mathscr{S}$ iff there exists $x \in J$ such that

$$h = \frac{\lim_{n \to \infty} n^{-1} \log \mu_n(x)}{\lim_{n \to \infty} n^{-1} \log \nu_n(x)}, \qquad \gamma = \frac{\lim_{n \to \infty} n^{-1} \log \lambda_n(x)}{\lim_{n \to \infty} n^{-1} \log \nu_n(x)}$$

Let $h(q, p) = \partial\tau(q, p)/\partial q$ and $\gamma(q, p) = \partial\tau(q, p)/\partial p$, where $\tau(q, p)$ is defined in Eq. (59). From Eq. (60), it is easy to show that

$$h(q, p) = \mu_{q,p,\tau(q,p)}(\varphi_\mu)/\mu_{q,p,\tau(q,p)}(\varphi_\nu)$$

$$\gamma(q, p) = \mu_{q,p,\tau(q,p)}(\varphi_\lambda)/\mu_{q,p,\tau(q,p)}(\varphi_\nu)$$

where $\mu(\varphi) = \int \varphi\, d\mu$.

In the Appendix we prove the following lemma:

Lemma 3. Unless the set $\mathscr{S}$ is trivial (i.e., a point or a segment), the function $(q, p) \to (h(q, p), \gamma(q, p))$ is invertible on the interior of $\mathscr{S}$ and its inverse is real analytic.

In the following, $q(h, \gamma)$ and $p(h, \gamma)$ will denote the unique values such that $h(q, p) = h$ and $\gamma(q, p) = \gamma$.

Let

$$\mathscr{F}'_{h,\gamma} = \left\{ x \in J,\ h = \frac{\lim_{n \to \infty} n^{-1} \log \mu_n(x)}{\lim_{n \to \infty} n^{-1} \log \nu_n(x)} \text{ and } \gamma = \frac{\lim_{n \to \infty} n^{-1} \log \lambda_n(x)}{\lim_{n \to \infty} n^{-1} \log \nu_n(x)} \right\}$$

From Lemma 2, we have for such sequences $h(x)=h$ and $\beta(x)=\gamma-1$ and then $\mathscr{F}'_{h,\gamma}\subset\mathscr{F}_{h,\gamma}$; we thus have $F'(h,\gamma)=\mathrm{Dim}_H(\mathscr{F}'_{h,\gamma})\leqslant F(h,\gamma)$. Let us show that

$$F'(h,\gamma)\geqslant\min_{q,p}\,(qh+p\gamma-\tau(q,p)) \tag{66}$$

Now, $\mathscr{S}$ is trivial if and only if $\log\lambda_i$ can be written as a linear combination of $\log\mu_i$ and $\log\nu_i$, i.e., $\exists(c_1,c_2)$ such that $\log\lambda_i=c_1\log\mu_i+c_2\log\nu_i$, $\forall i\in\{1\cdots s\}$ (this is always the case if $s=2$). In this case, one can directly apply Theorem 5 to get an estimate of $F'(h,\gamma)$ from the Legendre transform of $\tau(q,p)$ and Eq. (61) is proven.

Let us now suppose that $\mathscr{S}$ is nontrivial. The inequality (66) can be obtained using results from the thermodynamic formalism. Let $(h,\gamma)\in\mathscr{S}$ and let us consider the Gibbs state $\rho=\mu_{q(h,\gamma),p(h,\gamma),\tau(q,p)}$. One can show[40] that $\rho(\mathscr{F}'_{h,\gamma})=1$, so that we can directly apply the main theorem proved in ref. 45 to obtain

$$F'(h,\gamma)\geqslant s(\rho)/\chi(\rho) \tag{67}$$

where $s(\rho)$ is the metric entropy of ρ and $\chi(\rho)$ is the characteristic (Lyapunov) exponent $\chi(\rho)=\rho(\log|dT(x)/dx|)=-\rho(\varphi_\nu)$. We can then use the variational principle [Eq. (65)] and the fact that $P(q,p,\tau(q,p))=0$ to deduce that

$$s(\rho)/\chi(\rho)=q(h,\gamma)\,h+p(h,\gamma)\,\gamma-\tau(q(h,\gamma),p(h,\gamma))$$

In the Appendix we show that this last expression is nothing but $\min_{q,p}(qh+p\gamma-\tau(q,p))$, which achieves the proof of the inequality (63). ∎

As an illustration, we show in Fig. 6 the $\tau(q,p)$ and $D(h,\beta)$ spectra of the multifractal function $(\in\mathscr{N})$ described in Fig. 5a.

Remark. From the last theorem, one gets that the $D(h,\beta)$ singularity spectrum is a concave function. It follows that the support $\mathscr{S}$ of $D(h,\beta)$ is a convex set. One can easily prove that this set corresponds to the points $(x,y)\in\mathbb{R}^2$ such that $\exists(r_1,\ldots,r_s)\in[0,1]^s$, $\sum_i r_i=1$ with

$$x=\frac{\sum_{k=1}^{s} r_k\log\mu_k}{\sum_{k=1}^{s} r_k\log\nu_k},\qquad y=\frac{\sum_{k=1}^{s} r_k\log\lambda_k}{\sum_{k=1}^{s} r_k\log\nu_k}$$

Actually one can prove[40] that it corresponds to the convex envelope of the s-uple $\{(\log\mu_k/\log\nu_k,\,\log\lambda_k/\nu_k)\}_{1\leqslant k\leqslant s}$.

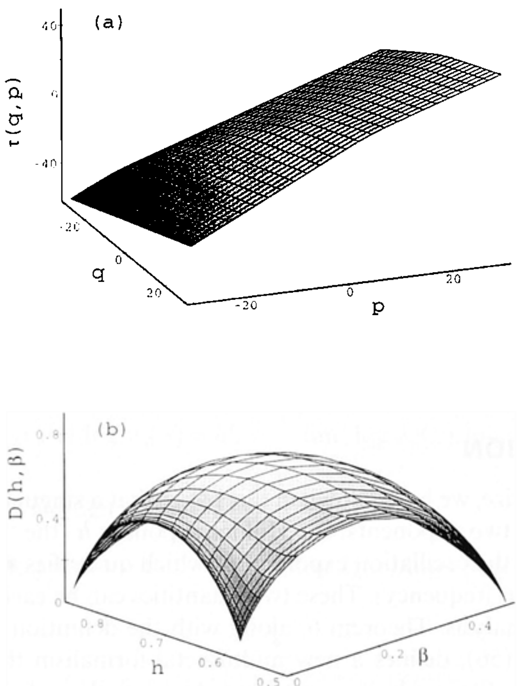

Fig. 6. Multifractal spectra of the singular distribution (ϵ, Γ) described in Fig. 5a. (a) $\tau(q, p)$ spectrum. (b) $D(h, \beta)$ spectrum.

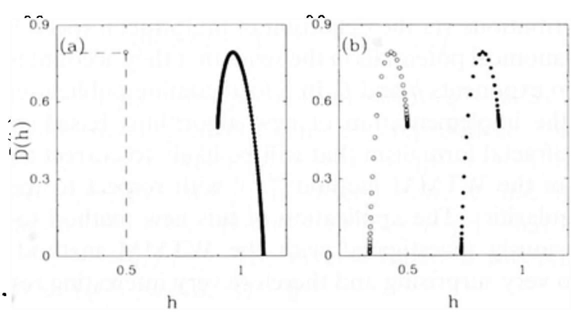

Fig. 7. (a) $D(h)$ singularity spectrum of the monofractal function described in Fig. 5b (◦) and of its ε-primitive ($\varepsilon = 0.4$, •), which is clearly multifractal because of the presence of oscillating singularities [Eq. (68)]. (b) The same $D(h)$ singularity spectra computed with the "classical" canonical multifractal formalism using the WTMM method. When oscillating singularities are present, the WTMM method leads to a wrong estimate of the $D(h)$ singularity spectrum and an ε-integration amounts to a simple shift of the spectrum.

Remark. Since β is linked to the derivative of h_ε at $\varepsilon = 0$ [Eq. (19)], one can prove[40] that the singularity spectrum $D_\varepsilon(h, \beta)$ of the ε-primitive of f is given by

$$D_\varepsilon(h, \beta) = D(h - \varepsilon(\beta + 1), \beta) \tag{68}$$

Thus, as shown in Fig. 7a, one can build some monofractal distributions [in the sense that $D(h)$ is supported by a single point] whose primitives are multifractal. Indeed, for instance, we suppose that for all k, we have $\log \mu_k = h_0 \log \nu_k$, and that $\lambda_k < \nu_k$. Then it is easy to check that all the singularities correspond to the same Hölder exponent $h(x) = h_0$, but with different values for $\beta(x)$. Thus, they are monofractal in terms of $h(x)$, but, because of the relation (68), they become multifractal when integrated.

5. CONCLUSION

To summarize, we have shown in this paper that a singular behavior must be described by two exponents: the Hölder exponent h (the "strength" of the singularity) and the oscillation exponent β (which quantifies the divergence of the instantaneous frequency). These two quantities can be easily characterized using wavelet analysis. Theorem 6, along with the definition of the partition function in Eq. (56), defines a new multifractal formalism that allows us to estimate the $D(h, \beta)$ singularity spectrum for a large class of singular distributions involving both cusp and oscillating singularities. Let us recall that the "classical" canonical formalism (Section 3) leads to a wrong $D(h)$ singularity spectrum if oscillating singularities are involved, as illustrated in Fig. 7b. This is the case, for instance, for the distributions in $\mathscr{N}$ that do not belong to $\mathscr{M}$. The newly defined multifractal formalism succeeds in characterizing statistically these distributions via the definition of multifractal spectra that play the role of grand-canonical potentials in the sense that they account for the fluctuations of the two exponents h and β. In a forthcoming publication, we hope to elaborate on the implementation of new algorithms based on this grand-canonical multifractal formalism that will be likely to correct for the intrinsic insufficiencies of the WTMM method[28–30] with respect to the detection of oscillating singularities. The application of this new method to experimental situations previously investigated with the WTMM method might occasionally lead to very surprising and therefore very interesting results.

APPENDIX

Lemma 3. Unless the set $\mathscr{S}$ is trivial (i.e., a point or a segment), the function $(q, p) \to (h(q, p), \gamma(q, p))$ is invertible on the interior of $\mathscr{S}$ and its inverse is real analytic.

Proof. Let us show that if $\mathscr{S}$ is nontrivial, the function

$$(q, p) \rightarrow (h(q, p) = \partial\tau(q, p)/\partial q, \gamma(q, p) = \partial\tau(q, p)/\partial p)$$

is invertible on the interior of $\mathscr{S}$. Let $(h, \gamma) \in \operatorname{int} \mathscr{S}$. Suppose $q > 0$ and $p > 0$. Then, $\exists \varepsilon_1, \varepsilon_2 > 0$ such that $(h' = h - 2\varepsilon_1,\ \gamma' = \gamma - 2\varepsilon_2) \in \mathscr{S}$. Let us rewrite the partition function $Z_n(q, p, \tau)$ as

$$\mathscr{Z}_n(q, p, \tau) = \sum_{\bar{k}_n} \exp[(-\tau + qh + p\gamma + q(\log \mu_{\bar{k}_n}/\log \nu_{\bar{k}_n} - h) + p(\log \lambda_{\bar{k}_n}/\log \nu_{\bar{k}_n} - \gamma)) \log \nu_{\bar{k}_n}]$$

where we have denoted by $\bar{k}_n$ the indices $k_1 \cdots k_n$. Let $x \in J$ such that

$$\lim_{n \to \infty} \log \mu_n(x)/\log \nu_n(x) = h', \qquad \lim_{n \to \infty} \log \lambda_n(x)/\log \nu_n(x) = \gamma'$$

Then for n large enough, we have

$$\log \mu_n(x)/\log \nu_n(x) - h < -\varepsilon_1, \qquad \log \lambda_n(x)/\log \nu_n(x) - \gamma < -\varepsilon_2$$

It follows that

$$\mathscr{Z}_n(q, p, \tau) \geqslant e^{(-\tau + qh - q\varepsilon_1 + p\gamma - p\varepsilon_2)\, n \log \nu}$$

where we have denoted by ν the greatest or the smallest value of ν_k [depending upon the sign of $-\tau + q(h - \varepsilon_1) + p(\gamma - \varepsilon_2)$]. Then, from Eq. (58), one has $P(q, p, \tau) \geqslant \log \nu(-\tau + q(h - \varepsilon_1) + p(\gamma - \varepsilon_2))$; since $\log \nu < 0$ and $P(q, p, \tau(q, p)) = 0$, one gets

$$qh + p\gamma - \tau(q, p) \geqslant \varepsilon_1 q + \varepsilon_2 p$$

Consequently, $-\tau(q, p) + qh + p\gamma \rightarrow +\infty$ when $q \rightarrow +\infty$ or $p \rightarrow +\infty$. The same kind of argument can be reproduced to show that $-\tau(q, p) + qh + p\gamma \rightarrow +\infty$ when $|q| \rightarrow +\infty$ or $|p| \rightarrow +\infty$. Since $\tau(q, p)$ is strictly concave, $qh + p\gamma - \tau(q, p)$ is strictly convex and thus admits a unique minimum at some point $(q(h, \gamma), p(h, \gamma))$ that satisfies $h = \partial\tau(q, p)/\partial q$ and $\gamma = \partial\tau(q, p)/\partial p$. The inverse function theorem ensures the real analyticity of $(q(h, \gamma), p(h, \gamma))$. ∎

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